Semiclassical estimation of the radiative mean lifetime τ of a hydrogen-like state

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The lifetime τ can be calculated quantum mechanically for a transition $n, l \to n', l'$, where $n' < n$ and $l' = l \pm 1$, for an H-like state. The calculation, however, can be messy for a particular transition from a state with large principal quantum numbers, say $n = 200$ and $l = 190$. Furthermore there can be literally hundreds of lower-lying levels, n', l'. Finally, there may be very many values of n and l for which one wishes to know τ. A semiclassical (SCl) estimate of τ, on the other hand, can be quite trivial and therefore quite useful. In addition, SCl theory can provide insights to results obtained by inspection of numerical values obtained quantum mechanically. One final point. One knows that SCl theory becomes accurate for large quantum numbers but little is known, theoretically, about the rate of convergence. In fact, one knows from experience that SCl results are generally rather accurate even for small quantum numbers. A statement attributed to J. C. Taylor describes the situation nicely: "The difference between large and small is small."

We begin by recalling the classical result for the power loss of a particle of mass m and charge e undergoing an acceleration $\mathbf{a}$,

$$\frac{dE}{dt} = -\frac{2}{3}\frac{e^2 a^2}{c^3}.$$

For the particle in a potential $V(r) = -Ze^2/r$, we have $a^2 = (Z^2e^4/m^2)/(1/r^4)$. For an initial state nl we take $1/r^4 =< 1/r^4 >_{nl}$, which is known. (Since the particle undergoes a huge number of orbits in the time τ, the orbit is well defined.) If then we replace dE/dt by $\Delta E/\tau$, we have $\tau = \Delta E f$, where f is known. The problem is that ΔE is not normally well defined, for, as noted above, there are normally many lower-lying levels. An exception occurs for $n, l = n-1$, for then the only allowed transition is to $n-1, l = n-2$. (For an electron or a muon in a hydrogenic state of given n — the muon might be inside the K shell of an electron — the time T to cascade to the ground state can be calculated since ΔE is then known, but T is not a mean lifetime.)

To estimate τ for the general case, we considered the classical result for the rate of change of angular momentum for an electron in a Coulomb field,

$$\frac{d\mathbf{L}}{dt} = -\frac{2\alpha^5 mc^2}{\hbar} Z\left(\frac{a_0}{r}\right)^3 \mathbf{L}, \tag{1}$$

where α is the fine-structure constant and a_0 is the Bohr radius. Since the classical motion is confined to a plane, which we can take to be the $x-y$ plane, we can replace $\mathbf{L}$ by $L_z = l\hbar$ and $d\mathbf{L}/dt$ by $dL_z/dt = \Delta L_z/\tau = \hbar/\tau$. Furthermore $1/r^3 \to < 1/r^3 >_{nl}$ which is known. We thereby arrive at a well-defined value of $1/\tau$. That's it. In the power-loss approach, there remained ΔE, generally not well defined; on the contrary, ΔL_z *is* well defined. τ is completely determined by the properties of the initial state; the final states need not be specified [156].

Note that Eq. (1), a classical equation, describes a *loss* of angular momentum. We therefore expect that, quantum mechanically, transitions $l \rightarrow l+1$ will be much weaker than transitions $l \rightarrow l-1$. This feature has been cited as a numerical observation; it is here given a trivial physical interpretation.

The SCl estimate of τ is good to at least one part in 10^4 for $n = 50, l = 49$ and is reasonably accurate even for small values of n and l.

The approach using Eq. (1) has been used to estimate the time it takes an electron to fall into the origin, and it seemed strange to us that it had not been used to estimate τ. In fact, as we learned when we were at the galleys stage, it had been used and the results published, but in a somewhat unknown journal. Our treatment was, we believe, slightly better.

A similar technique was used to estimate τ for an electron in a Landau bound state in a uniform magnetic field [157].