New York University

Physics Department

PRELIMINARY EXAMINATION FOR THE PH.D. DEGREE

DYNAMICS

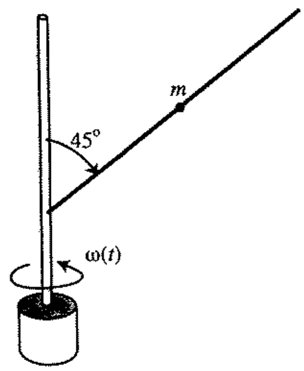
Fall, 2007

READ INSTRUCTIONS CAREFULLY

1. ANSWER ALL OF THE PROBLEMS.
2. You have 3 hours to complete the examination.
3. Use a separate answer booklet for each problem. On the front cover of each booklet write the problem number and your own identification number.
4. Show ALL your work.

Problem I (29 pts)

A bead of mass m moves without friction on a straight wire inclined at an angle of 45^o to the vertical, as shown in the figure. An electric motor maintains an angular velocity of $\omega(t)$ about the vertical axis, independent of the state of the bead. Gravity acts downward with acceleration g.



(a) (7 pts) How many degrees of freedom does the system have? Choose an appropriate generalized coordinate system for the configuration space. The lower endpoint of the wire (point of attachment to the axle) should be chosen as the origin. Write down the Lagrangian for the motion of the bead.

(b) (7 pts) Make a change of variables and obtain the Hamiltonian for the motion of the bead. Is the Hamiltonian conserved? Explain. Is the Hamiltonian the total energy of the bead? Explain.

(c) (5 pts) Determine all fixed points of the bead in the chosen coordinate system. In each case, indicate whether the fixed point is stable, unstable , or neutral.

(d) (10 pts) For constant ω, sketch the phase portrait, showing phase-space orbits which , for different initial conditions,

(A) remain fixed at a certain distance from the origin,

(B) start near the origin, move outward toward an equilibrium point, then return to the origin,

(C) start near the origin, move outward to infinity,

(D) start far from the origin, move inward, and eventually reach the origin,

(E) start far from the origin, move inward with decreasing speed, approaching, but never reaching the fixed point.

In your phase portrait, label each type of orbit with the relevant letter (A,B,.....).

Problem II (29 pts)

A perturbed harmonic oscillator has the Hamiltonian

$$H(q,p) = \frac{1}{2}(p^2 + \omega^2 q^2) + \epsilon q^2 p.$$

(a) (14 pts) Find a canonical change of variables from (q,p) to (Q,P) such that

$$P = \frac{1}{2\omega}(p^2 + \omega^2 q^2).$$

Suggestion: guess the answer and verify the required Poisson bracket relations, or else derive the answer using a generating function.

(b) (5 pts) What is the new Hamiltonian, $K(Q,P)$?

(c) (10 pts) Find a new canonical transformation, from (Q,P) to $(\bar{Q},\bar{P})$, such that the transformed Hamiltonian takes the form

$$\bar{K}(\bar{Q},\bar{P}) = \bar{K}_0(\bar{P}) + O(\epsilon^2).$$

This should be a perturbative calculation: at each step it is necessary to keep only terms up to first order in ϵ.

Problem III (8 pts)

Consider a dynamical system with Hamiltonian $H(x,y,p_x,p_y)$. Suppose that in a region $\mathcal{S}$ of phase space, there exists a polynomial function $K(x,y,p_x,p_y)$ such that H and K are functionally independent and have vanishing Poisson bracket with one another. Let $\mathcal{M}_{h,k}$ be a surface on which H and K assume the fixed values h and k, respectively. Suppose that each such surface is compact (closed and bounded).

(a) (3 pts) What is the topological structure of each surface $\mathcal{M}_{h,k}$? (This is a consequence of the Liouville-Arnol'd theorem.)

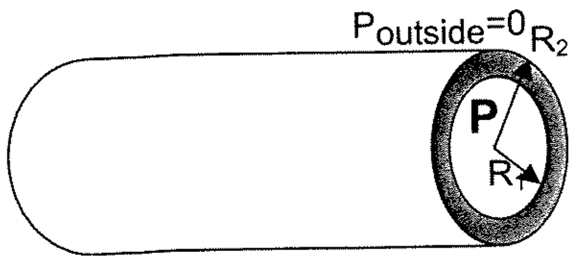
(b) (3 pts) If a canonical transformation to action-angle variables $(\theta_1, \theta_2, J_1, J_2)$ is made on $\mathcal{S}$, what is the form of the new Hamiltonian $\bar{K}(\theta_1, \theta_2, J_1, J_2)$?

(c) (2 pts) What is the time-dependence of the angle variables, assuming that $\bar{K}$ has been determined?

Some possibly useful formulas for Problems IV and V may be found at the end of the examination.

Problem IV (17 pts)

Stress in a pressurized cylindrical pipe. A solid cylindrical shell of inner radius R_1 and outer radius R_2, thickness R_2-R_1, contains a gas at pressure P. The pressure outside is zero. Assume that the cylinder is capped in such a way that the strain along z is zero. Calculate the stress and strain profile e.g. σ_{rr}, σ_{zz}, $\sigma_{\theta\theta}$.



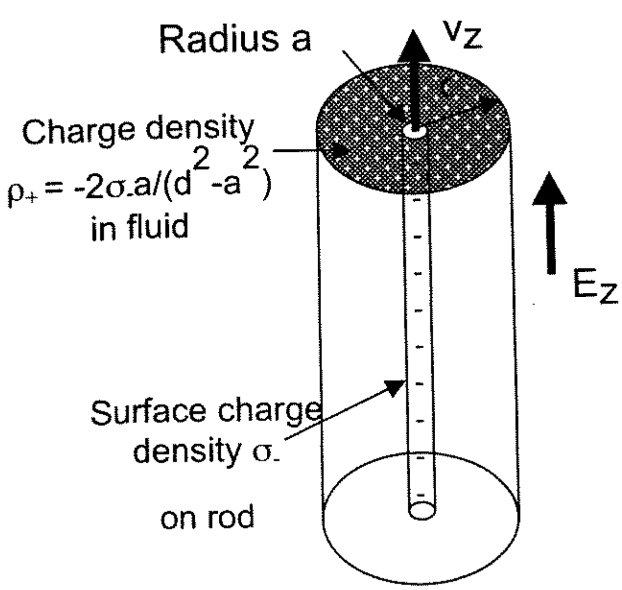
Problem V (17 pts)

Electro-osmosis, Electrophoresis. A long cylinder, radius d, contains a viscous charged fluid with viscosity η and constant charge density $\rho_+ = -2(\sigma_-)a/(d^2-a^2)$. At the center of the cylinder is a long thin rod of radius a with surface charge σ_-, so that the entire system is charge neutral. The outer cylinder is fixed and does not move. An electric field, E_z, is applied axially throughout the cylinder.

a) Assuming low Reynolds number find the velocity in the fluid, $v_z(r)$. (10 pts.)

b) Find the velocity of the thin charged rod. (3 ps.)

c) Find the stresses on the surfaces of the outer cylinder and the charged rod. (4 pts.)



Cylindrical polar coordinates (R, θ, z)

$$\nabla\Phi = \frac{\partial\Phi}{\partial R}\hat{\mathbf{R}} + \frac{1}{R}\frac{\partial\Phi}{\partial\theta}\boldsymbol{\theta} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{R}\frac{\partial}{\partial R}(RF_R) + \frac{1}{R}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$$

$$\nabla\times\mathbf{F} = \Big[\frac{1}{R}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\Big]\hat{\mathbf{R}} + \Big[\frac{\partial F_R}{\partial z} - \frac{\partial F_z}{\partial R}\Big]\hat{\theta} + \frac{1}{R}\Big[\frac{\partial}{\partial R}(RF_\theta) - \frac{\partial F_R}{\partial\theta}\Big]\hat{\mathbf{z}}$$

$$\nabla^2\Phi = \frac{1}{R}\frac{\partial}{\partial R}\Big(R\frac{\partial\Phi}{\partial R}\Big) + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\mathbf{F} = \Big[\nabla^2F_R - \frac{1}{R^2}F_R - \frac{2}{R^2}\frac{\partial F_\theta}{\partial\theta}\Big]\hat{\mathbf{R}}$$
$$+\Big[\nabla^2F_\theta - \frac{1}{R^2}F_\theta + \frac{2}{R^2}\frac{\partial F_R}{\partial\theta}\Big]\hat{\boldsymbol{\theta}} + \nabla^2F_z\,\hat{\mathbf{z}}$$

$$(\mathbf{B}\cdot\nabla)\mathbf{A} = [\mathbf{B}\cdot\nabla A_R - B_\theta A_\theta/R]\hat{\mathbf{R}}$$
$$+[\mathbf{B}\cdot\nabla A_\theta + B_\theta A_R/R]\hat{\theta} + \mathbf{B}\cdot\nabla A_z\,\hat{\mathbf{z}}$$

$$u_{rr} = \frac{\partial u_r}{\partial r}, \quad u_{\phi\phi} = \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r}, \quad u_{zz} = \frac{\partial u_z}{\partial z},$$

$$2u_{\phi z} = \frac{1}{r}\frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z}, \quad 2u_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r},$$

$$2u_{r\phi} = \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \phi}.$$

$$\mu = E/2(1+\sigma), \; K = E/3(1-2\sigma). \tag{5.9}$$

We shall write out here the general formulae of §4, with the coefficients expressed in terms of E and σ. The free energy is

$$F = \frac{E}{2(1+\sigma)}\left(u_{ik}^2 + \frac{\sigma}{1-2\sigma}u_{ll}^2\right). \tag{5.10}$$

The stress tensor is given in terms of the strain tensor by

$$\sigma_{ik} = \frac{E}{1+\sigma}\left(u_{ik} + \frac{\sigma}{1-2\sigma}u_{ll}\delta_{ik}\right). \tag{5.11}$$

Conversely,

$$u_{ik} = [(1+\sigma)\sigma_{ik} - \sigma\sigma_{ll}\delta_{ik}]/E. \tag{5.12}$$

$$\frac{E}{2(1+\sigma)}\frac{\partial^2 u_i}{\partial x_k^2} + \frac{E}{2(1+\sigma)(1-2\sigma)}\frac{\partial^2 u_l}{\partial x_i \partial x_l} + \rho g_i = 0. \tag{7.1}$$

These equations can be conveniently rewritten in vector notation. The quantities $\partial^2 u_i/\partial x_k^2$ are components of the vector $\triangle \mathbf{u}$, and $\partial u_l/\partial x_l \equiv \operatorname{div} \mathbf{u}$. Thus the equations of equilibrium become

$$\triangle \mathbf{u} + \frac{1}{1-2\sigma}\,\mathbf{grad}\operatorname{div}\mathbf{u} = -\rho \mathbf{g}\frac{2(1+\sigma)}{E}. \tag{7.2}$$

It is sometimes useful to transform this equation by using the vector identity $\mathbf{grad}\operatorname{div}\mathbf{u} = \triangle\mathbf{u} + \mathbf{curl}\,\mathbf{curl}\,\mathbf{u}$. Then (7.2) becomes

$$\mathbf{grad}\operatorname{div}\mathbf{u} - \frac{1-2\sigma}{2(1-\sigma)}\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = -\rho\mathbf{g}\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}. \tag{7.3}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -\frac{1}{\rho}\mathbf{grad}\,p + \frac{\eta}{\rho}\triangle\mathbf{v}.$$

$$\begin{aligned}
\sigma_{rr} &= -p + 2\eta\frac{\partial v_r}{\partial r}, & \sigma_{r\phi} &= \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right),\\
\sigma_{\phi\phi} &= -p + 2\eta\left(\frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r}\right), & \sigma_{\phi z} &= \eta\left(\frac{\partial v_\phi}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \phi}\right),\\
\sigma_{zz} &= -p + 2\eta\frac{\partial v_z}{\partial z}, & \sigma_{zr} &= \eta\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right).
\end{aligned} \qquad (15.15)$$

The three components of the Navier–Stokes equation and the equation of continuity are

$$\begin{aligned}
&\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\phi^2}{r}\\
&\qquad = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \phi^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} - \frac{v_r}{r^2}\right),\\
&\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} + v_z\frac{\partial v_\phi}{\partial z} + \frac{v_r v_\phi}{r}\\
&\qquad = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \nu\left(\frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{\partial^2 v_\phi}{\partial z^2} + \frac{1}{r}\frac{\partial v_\phi}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2}\right),\\
&\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_z}{\partial \phi} + v_z\frac{\partial v_z}{\partial z}\\
&\qquad = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \phi^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right),\\
&\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0.
\end{aligned} \qquad (15.16)$$

Electromagnetism (Prelim, 3 hours)

PROBLEMS:

1. A $E = 10^{20}$eV proton (a cosmic ray) collides with a photon of energy ϵ at 90°. Calculate the threshold energy ϵ for the neutral pion production, $p + \gamma \rightarrow p + \pi^0$. ($m_p = 938$MeV, m_π=135 MeV.)

2. An electric dipole d is at a distance r from the surface of an infinite plane conductor. The dipole moment is perpendicular to the surface. Calculate the force F.

3. A parallel plate capacitor has the capacitance C_0 when the space between the plates is vacuum. One inserts two plane-parallel dielectric layers into the capacitor. The layers are of equal thickness, and they fill all the space between the plates of the capacitor. The permittivities of the layers are ϵ_1 and ϵ_2. Calculate the new capacitance C.

4. There is a charge Q and a dipole $\mathbf{d}$. The radius vector from the charge to the dipole is $\mathbf{r}$. Calculate the force $\mathbf{F}$ on the dipole.

5. The resistivity of aluminum is $\rho_1 = 2.7 \times 10^{-8}\Omega\cdot$m, of copper — $\rho_2 = 1.7 \times 10^{-8}\Omega\cdot$m. There is a straight wire of uniform cross section, which carries a current $I = 10$A. The left part of the wire is made of aluminum, the right part is made of copper. Calculate the charge Q which must sit at the contact surface due to this current. Give your numerical answer as Q/e, where e is the charge of electron. (For numerical answer you might need: $e = 1.6 \times 10^{-19}$C $= 4.8 \times 10^{-10}$cgs, $\Omega = V/A$, $V = 1/300$cgs, $A = 3 \times 10^9$cgs, $\epsilon_0 = 8.85 \times 10^{-12} C^2/N/m^2$.).

6. An ultra-relativistic charged particle (mass m, charge e, Lorentz factor $\gamma \gg 1$) is moving in the equatorial plane of a heavy (not moving) magnetic dipole μ. The impact parameter b is so large, that the particle is only slightly deflected when it passes near the dipole. Estimate the deflection angle χ.

7. A nonrelativistic charged particle (mass m, charge e, velocity v) is moving in the equatorial plane of a heavy magnetic dipole μ. The impact parameter b is so large, that the particle is only slightly deflected when it passes by the dipole. Estimate the radiated energy E.

8. Estimate the radiated energy E for the head on collision of a proton and an alpha particle. Charge $\sim e$, mass $\sim m$, collision velocity $\sim v$.

9. A flat piece of aluminum foil of small thickness δ and of area A is at a distance $R \gg \sqrt{A}$ from a point charge Q. The foil surface is perpendicular to the foil-charge radius vector. Calculate the force on the foil F.

Quantum Mechanics I **Preliminary Exam** **Fall 2007**

Wednesday August 29, 2007, 2:00 pm

Closed book. In many instances you can do later parts of a problem without doing all the preceding parts, so if one part of a problem seems non-obvious or looks long, consider skipping on and returning to it later. Do read the problems very carefully! Also, make your reasoning clear. All problems carry equal weight.

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1 eV $= 1.6 \times 10^{-19}$ J.
$e = 1.6 \times 10^{-19}$ C
$\hbar = 10^{-34}$ J s
$m_e = 9 \times 10^{31}$ Kg$= 0.5$ MeV$/c^2$

Some "hydrogen-like" radial wavefunctions, R_{nl}:

$R_{10}(r) = 2\left(\frac{Z}{a_o}\right)^{\frac{3}{2}} e^{-\frac{Zr}{a_o}}$

$R_{20}(r) = 2\left(\frac{Z}{2a_o}\right)^{\frac{3}{2}} \left(1 - \frac{Zr}{2a_o}\right) e^{-\frac{Zr}{2a_o}}$

$R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{Z}{2a_o}\right)^{\frac{3}{2}} \frac{Zr}{a_o} e^{-\frac{Zr}{2a_o}}$

Ground state energy of Hydrogen: $E_n = -\dfrac{e^4 m}{2\hbar^2} = -13.6\ eV$

Spherical harmonics and Clebsch Gordan coefficients are attached.

Problem 1: In a three-dimensional vector space, an observable A is represented by the following matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{1}$$

a) What are the possible outcomes of an ideal (projective) measurement of the observable A?

b) Find the probabilities of each of these possible measurement outcomes if A is measured on a system prepared in the state

$$|\psi_i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} . \tag{2}$$

c) For each of the possible measurement outcomes, find the final state of the system.

d) If, instead of Eq. (2), the system were prepared in the state defined by the density operator

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{3}$$

what would be the resulting state after each measurement outcome?

e) Are the states found in part (d) the same as those found in part (c)? Explain.

Problem 2:

a) Consider a potential given by

$$V(x) = -\frac{\hbar^2}{2m} c\, \delta(x), \tag{4}$$

for positive c. Find the bound state wavefunctions and energies for this potential. How many bound states are there?

b) Now consider the potential

$$V(x) = -\frac{\hbar^2}{2m} c\, [\delta(x+a) + \delta(x-a)] \tag{5}$$

How many possible bound states are there for this potential? Estimate (or calculate exactly) how the number of bound states depends on the parameters of the potential (c and a)?

c) Make a sketch of all the possible bound states, and indicate which state has the lowest energy.

Problem 3: Consider two *non*-identical particles of mass m and spin $1/2$. They interact only through the potential

$$V = \frac{g}{r}\vec{\sigma}_1 \cdot \vec{\sigma}_2\,, \tag{6}$$

where r is the distance between the two particles, $g > 0$, and $\vec{\sigma}_j$ are the Pauli spin matrices which operate on the spin of particle j. Express the potential V in terms of the total spin $\vec{S} = \vec{s}_1 + \vec{s}_2$, and find the energies of all the bound states.

Problem 4: The following Hydrogen atom matrix elements either vanish due to a selection rule, can be easily evaluated exactly using basic facts you should know, or can be *estimated.* Give your best estimate for each and *explain how you arrived at it.* (Of course all cases can be evaluated by integrating over the wavefunctions with the appropriate weight, but that is not expected here.)

a) $\langle 1s|\mathbf{R}^2|1s\rangle$

b) $\langle 2s|\mathbf{L}^2|2p\rangle$

c) $\langle 2s|\frac{\mathbf{P}^2}{2m_e}|2s\rangle$

d) $\langle 2s|\mathbf{Z}|2p, m = 0\rangle$; the angular momentum quantization axis is $+\hat{z}$ and $\mathbf{Z}$ is the z-component of the position operator.

Problem 5: Consider a simple harmonic oscillator with natural frequency ω in initial state

$$|\psi(0)\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\,, \tag{7}$$

where states $|n\rangle$ are defined by $N|n\rangle = n|n\rangle$, with $N = a^\dagger a$, and $a^\dagger$ and a are the harmonic oscillator raising and lowering operators.

a) Write down an expression for $|\psi(t)\rangle$.

b) What is the expectation value for the energy of this state?

c) Find the time dependence of the expectation value of $q = (a + a^\dagger)/\sqrt{2}$, where q is the unitless position operator.

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

1/2×1/2

m_1	m_2	1, +1
+1/2	+1/2	1

m_1	m_2	1, 0	0, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1, −1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2, +3/2
+1	+1/2	1

m_1	m_2	3/2, +1/2	1/2, +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2, −1/2	1/2, −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2, −3/2
−1	−1/2	1

2×1/2

m_1	m_2	5/2, +5/2
+2	+1/2	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2, +1/2	3/2, +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2, −1/2	3/2, −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2, −3/2	3/2, −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2, −5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2, +2
+3/2	+1/2	1

m_1	m_2	2, +1	1, +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2, 0	1, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2, −1	1, −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2, −2
−3/2	−1/2	1

2×1

m_1	m_2	3, +3
+2	+1	1

m_1	m_2	3, +2	2, +2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3, +1	2, +1	1, +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3, 0	2, 0	1, 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3, −1	2, −1	1, −1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3, −2	2, −2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3, −3
−2	−1	1

1×1

m_1	m_2	2, +2
+1	+1	1

m_1	m_2	2, +1	1, +1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2, 0	1, 0	0, 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2, −1	1, −1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2, −2
−1	−1	1

3/2×1

m_1	m_2	5/2, +5/2
+3/2	+1	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2, −3/2	3/2, −3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2, −5/2
−3/2	−1	1

3/2×3/2

m_1	m_2	3, +3
+3/2	+3/2	1

m_1	m_2	3, +2	2, +2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	3, +1	2, +1	1, +1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	3, 0	2, 0	1, 0	0, 0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	3, −1	2, −1	1, −1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	3, −2	2, −2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	3, −3
−3/2	−3/2	1

$$d^0_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

m_1	m_2	7/2, +7/2
+2	+3/2	1

m_1	m_2	7/2, +5/2	5/2, +5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	7/2, +3/2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	7/2, +1/2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	7/2, −1/2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	7/2, −3/2	5/2, −3/2	3/2, −3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	7/2, −5/2	5/2, −5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	7/2, −7/2
−2	−3/2	1

2×2

m_1	m_2	4, +4
+2	+2	1

m_1	m_2	4, +3	3, +3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	4, +2	3, +2	2, +2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	4, +1	3, +1	2, +1	1, +1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	4, 0	3, 0	2, 0	1, 0	0, 0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	4, −1	3, −1	2, −1	1, −1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	4, −2	3, −2	2, −2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	4, −3	3, −3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	4, −4
−2	−2	1

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

Quantum Mechanics II **Preliminary Exam** **Spring 2007**

Glennys Farrar and Tycho Sleator

Date and time of exam

Read the problems very carefully! Explain all steps and make your reasoning clear. In some instances you can do later parts of a problem without doing all the preceding parts, so if one part of a problem seems non-obvious or looks long, consider skipping on and returning to it later.

Problem 1: A particle of charge e and mass m is confined to move on the circumference of a circle of radius r. The only term in the Hamiltonian is the kinetic energy, so the eigenfunction and eigenvalues are

$$\psi_n(\phi) = \frac{1}{\sqrt{2\pi}} e^{in\phi}$$
$$E_n = \frac{\hbar^2 n^2}{2mr^2}$$

where ϕ is the angle around the circle. An electric field $\mathbf{E}$ is imposed in the plane of the circle.
(a) Find the perturbed energy levels up to $O(|\mathbf{E}|^2)$.
(b) Describe qualitatively the probability distribution as a function of angle for the perturbed and unperturbed states in the different energy levels.

Problem 2: (a) For a system of two separated spin 1/2 particles prepared in a singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_1 |\downarrow\rangle_2 - |\uparrow\rangle_1 |\downarrow\rangle_2) \ , \tag{1}$$

find the correlation function $C(\hat{a}, \hat{b}) \equiv \langle (\vec{\sigma}_1 \cdot \hat{a})(\vec{\sigma}_2 \cdot \hat{b}) \rangle$ for the state $|\psi\rangle$ as a function of the angle θ between unit vectors $\hat{a}$ and $\hat{b}$.

(b) What are the assumptions underlying the Bell inequality

$$|C(\hat{a}, \hat{b}) - C(\hat{a}, \hat{b}')| + |C(\hat{a}', \hat{b}') + C(\hat{a}', \hat{b})| \leq 2. \tag{2}$$

(c) Show that the correlation function $C(\hat{a}, \hat{b})$ applied to the state $|\psi\rangle$ violates the Bell inequality, Eq. (2) for appropriate choices of $\hat{a}$, $\hat{a}'$, $\hat{b}$, and $\hat{b}'$. What are the orientations of $\hat{a}$, $\hat{a}'$, $\hat{b}$, and $\hat{b}'$ that do this?

Problem 2b:

In Mermin's 3-particle version of "Bell's inequality without inequalities", three particles are prepared in a entangled spin state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle) \ , \tag{3}$$

where $\sigma_z| \uparrow\rangle = | \uparrow\rangle$ and $\sigma_z| \downarrow\rangle = -| \downarrow\rangle$. Each of the three particles is sent to one of three observers. Each observer then measures either σ_x or σ_y. In what follows, σ_{ix} and σ_{iy} represent measurements by the i'th observer on the i'th particle of σ_x and σ_y, respectively.

a) Show that $|\psi\rangle$ is an eigenstate of both $\sigma_{1x}\sigma_{2x}\sigma_{3x}$ and $\sigma_{1y}\sigma_{2y}\sigma_{3x}$. What are the eigenvalues?
b) Based on your results of part a), what is the EPR (Einstein, Podolsky, Rosen) type argument that leads one to define "elements of reality" in this situation, and what are these "elements of reality"?
c) What are the assumptions being made about these elements of reality that lead to an incompatibility with the predictions of quantum mechanics?

Problem 3: The strong interactions of nucleons and mesons are described by a Hamiltonian which is invariant under isospin rotations and conserves charge. The proton and neutron form an isospin doublet (proton having $I_z = +\frac{1}{2}$ and neutron having $I_z = -\frac{1}{2}$). There are three pions (also called pi-mesons) which form an isospin triplet π^+, π^0, π^- having $I_z = +1,\ 0, -1$ respectively.
When a pion and a nucleon collide at certain energies, a *resonance* can be formed. Resonances are short lived states which have fairly well-defined masses; they are unstable and decay, in this case mostly to a nucleon and pion.

a) It is observed that a particular resonance decays 2/3 of the time to $\pi^0\ n$ and 1/3 of the time to $\pi^-\ p$. What is the value of $|I,\ I_z\rangle$ of this resonance?

b) The resonance in part a) is a member of a multiplet. How many *other* members does this multiplet have?

c) What are the charges of this and the other members of the multiplet?

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

2×3/2

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	J=7/2, M=−7/2
−2	−3/2	1

3/2×3/2

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=−2	J=2, M=−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	J=3, M=−3
−3/2	−3/2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

2×2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	J=4, M=−3	J=3, M=−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	J=4, M=−4
−2	−2	1

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta - 1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta + 1}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1) \qquad d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

STATISTICAL PHYSICS
(PRELIM.)

1. The fluid has a slab geometry and can be compressed along x keeping the constant cross section S. Considering the adiabatic process of compressions and rarefactions, show that the velocity of sound propagation

$$u^2 = \gamma p_0 / \rho_0$$

where p_0 is equilibrium pressure and ρ_0 is equilibrium density of the fluid and γ is adiabatic constant.

(2 points)

2. The free expansion of a gas is a process where the total mean energy E remains constant. Find $(\partial T / \partial V)_E$ and calculate the temperature change $\Delta T = T_2 - T_1$ in a free expansion of a van der Waals gas assuming c_V to be temperature independent.

(3 points)

3. A very sensitive spring balance consists of a quartz spring suspended from a fixed support. The spring constant is α, i.e., the restoring force of the spring is $-\alpha x$ if the spring is stretched by an amount x. The balance is at a temperature T in a location where the acceleration due to gravity is g.

(a) If a very small object of mass M is suspended from the spring, what is the mean resultant elongation x of the spring?

(b) What is the magnitude $\overline{(x-\bar{x})^2}$ of the thermal fluctuations of the object about its equilibrium position?

(c) It becomes impracticable to measure the mass of an object when the fluctuations are so large that $[\overline{(x-\bar{x})^2}]^{\frac{1}{2}} - x$. What is the minimum mass M which can be measured with this balance?

(2 points)

4. Photons are contained in a cubical box with sides L at equilibrium state of temperature T. Write the distribution of photons with respect to their energy and expression for the pressure of photons (radiation pressure). How does this expression differ for non-relativistic free particles in a cubical box?

(3 points)

5. Derive the expression for entropy S of the system of Bose-Einstein and Fermi-Dirac particles using the distributions $\bar{n}_s$ for the number of particles at the state with energy ϵ_s and the partition function z

$$\ln Z = \alpha N \pm \sum_s \ln(1 \pm e^{-\alpha - \beta\epsilon_s})$$

Explain the origin of this partition function. What is α?

(2 points)

6. For degenerate ideal Fermi gas and Bose gas find the fluctuation of the number of particles at the state k, $\langle(\Delta n_k)^2\rangle$, within a fairly small volume. Apply the result to obtain the fluctuation of the energy $\Delta E_{\Delta\omega}$ of the photons energy $E_{\Delta\omega}$ within small frequency interval $\Delta\omega$ near the ω_k. Assume the photons are from the black body radiation.

(3 points)