Although I have known John Duff and his work for many years, I first had contact with him in a semi-scientific manner a little over a decade ago, when he contacted me to inform himself about Penrose tiles, a geometric exploration of the transition from periodic order to disorder. What mathematicians and physicists realized in the 1970s and 1980s, is that between the order of a periodic crystal and an amorphous (disordered) structure, there is something called ‘quasi-order’, with regularity but no single spatial period. In 1984 metallurgists succeeded in synthesizing a material with such a structure experimentally, even as theoretical physicists independently predicted their possibility. Figure 1 shows different two-dimensional (planar) arrays, going from periodic tiling (a), to quasiperiodic order generated by two different tiles (b), to disorder characteristic of amorphous materials (right-hand end of part c).

In this new exhibition John Duff continues to use his artistic inspiration and intuition to explore the geometry of space, and in this way he comes into contact with contemporary developments in science, some of which I would like to describe briefly below. In addition to the transition between order and disorder, these include packing problems and the partitioning of space, the effects of dimensionality and curvature, and how frustration can lead to disorder.

In applying these concepts to natural phenomena it is important to understand the spatial scale at which the phenomena manifest themselves. Let us distinguish three different scales: the nanoscale, 1 nanometer (nm) = one billionth of a meter = $10^{-9}$ m. This is the scale of atomic and molecular phenomena. The next scale is the mesoscale, 1 micrometer ($\mu$m) = one millionth of a meter = $10^{-6}$ m, the scale of electronic microchips, colloids or living cells. Finally, we have the macroscale of human experience, which extends anywhere from submillimeter
(1 millimeter = $10^{-3}$ m = one thousandth of a meter) to many kilometers (1 kilometer = one thousand meters = $10^{+3}$ m). [We have not used the term ‘microscale’ because its meaning is ambiguous. Logically, it should refer to the scale of microns (μm), e.g. what is seen in a light microscope, which we have called the ‘mesoscale’. In scientific parlance, however, the terms ‘microscale’ or ‘microscopic description’ refer to phenomena on the scale of atoms and molecules, i.e. what we have called the ‘nanoscale’].

An instructive way to think about the different scales comes from electromagnetic waves, which can propagate (always with the same velocity in vacuum, $c = 10^8$ m/second) with wavelength at any scale. At the nanoscale the waves are x-rays and ultraviolet light, at the mesoscale they are visible light and at the macroscale the waves go from microwaves ($10^{-2}$ m) to radio waves (1 to $10^2$ m).

The traditional methods of materials synthesis involve manipulating atoms and molecules, and thus determining structure primarily on the nanoscale. The study of materials whose properties are determined by their structure on the mesoscale is a relatively recent scientific and technological development, and it has shown significant progress in recent years, under the name of ‘soft matter physics and chemistry’. I will attempt to illustrate below some of the phenomena where the ideas explored by John Duff become relevant for soft matter research.

**Packing of hexagons and tetrahedra**

The lattice of hexagons in Fig. 1a works well in flat space (two dimensions) but if it is attempted in curved space, e.g. on the surface of a sphere, one encounters frustration, meaning that the hexagons will not tile the surface without creating voids. It turns out, however, that a set of 15 hexagons fits perfectly if 12 pentagons are interspersed in such a way that each hexagon is adjacent to three other hexagons and three pentagons, leading to the familiar soccer ball pattern shown in Figure 2.

Figure 2. A soccer ball with 15 hexagons and 12 pentagons

Figure 3. The icosahedron, with 20 triangular faces and 12 vertices
The generalization of a two-dimensional triangle to three dimensions is the tetrahedron, one of the 5 platonic solids, made up of 4 equilateral triangles. Another platonic solid is the icosahedron, pictured in Figure 3, which has 20 equilateral triangles for faces and 12 vertices. One can draw a sphere through the vertices and attempt to construct tetrahedra by joining the vertices of the icosahedron to the center of the circle, thus filling the space inside the sphere with 20 ‘tetrahedra’. It turns out, however, that the radius of the circle is equal to $0.951e$ rather than $e$, where $e$ is the length of the edge, so the triangles formed by joining an edge to the center of the sphere are not equilateral. Instead, they are isosceles triangles and the 20 four-sided polyhedra are actually ‘triangular pyramids’ with an equilateral base and three isosceles sides, not regular tetrahedra. This situation is another example of frustration.

John Duff has chosen to experience this same frustration in a different manner. He starts out by fixing the building blocks to be regular tetrahedra, and then packs them together attempting to form the icosahedron. As we saw above this is impossible, so he ends up with what he calls the ‘fractured icosahedron’ shown in Figure 4, with voids between the tetrahedra, filling what one can call the ‘inverse space’.

**Figure 4** The fractured icosahedron by John Duff

**Figure 5** Sphere packing in hcp lattice

**Sphere packing**

In addition to tetrahedral, John Duff has explored many aspects of geometry arising from the packing of spheres. It has been known for millenia that efficient ways to do this are the regular face centered cubic (fcc) or the hexagonal close packed (hcp) lattices used to stack fruit, for example as shown in Figure 5. In the 17th century the great mathematician and astronomer Johannes Kepler, conjectured that this lattice was the most efficient way to pack equal sized spheres, in the sense that the voids would occupy the smallest fraction of the total volume. It
was soon realized that the conjecture was correct under the assumption that the spheres are in an ordered array, a lattice, but it was not until a few years ago that the Kepler conjecture was proved for *disordered* arrays of equally sized spheres as well.

John Duff’s fascination with the Kepler conjecture focuses on what we call the ‘inverse space’, namely the voids between the spheres. In the efficient packings we are considering, the spheres touch their neighbors, so one can imagine a ‘direct space’ consisting of the connected volumes of all the spheres. The inverse space between the spheres is also a connected volume extending over the whole body. The artist has brought this inverse space to the fore in his constructions by filling it with a plastic material which solidifies, whereupon he is able to remove the spheres to reveal the inverse space by itself, which he calls ‘Inside the Kepler conjecture’, as shown in Figure 6, one of many such constructions in the present exhibit. Before removing the spheres the contact point between them was replaced by a ring, which remained when the spheres were removed, thus making the direct space occupied by the spheres easier to imagine. The sculpture consists entirely of elements of the inverse space. As mentioned, there are many other examples of such inverse spaces in the exhibit, most of which are formed by stitching together different lattices of spheres at interesting angles, and then removing the spheres and filling in the inverse space.

Figure 6. Inside the Kepler conjecture by John Duff
Optical lattices
A scientific goal which has been pursued for a number of decades is the construction of so-called ‘optical lattices’. These are transparent materials whose periodicity is on the same scale as the wavelength of light (0.6 μm), as opposed to crystal lattices with nanoscale periodicity. A particular project has been to build a structure analogous to the one in Figure 6, but where both the direct and inverse spaces are filled with transparent materials whose light propagation properties (index of refraction) are different. If light at a certain wavelength only propagates in one of the materials, for example, such a device would allow particularly fast and accurate manipulation of light rays, and might in fact lead to optical computing. The inverse space of these optical lattices goes by the name of the ‘inverse opal’, because opals are precisely made up of periodic arrays of microspheres. Devices with the above properties have not yet been made practical, but they hold great promise and they illustrate how the same interpenetration of a direct and inverse space as the ones explored by John Duff are relevant to modern technology.

Plumber’s nightmare
Other areas of scientific research where mesoscale structures with direct and inverse spaces have been encountered are in the synthetic chemistry of polymer-ceramic composites or of microemulsions. If one imagines that the contiguous but separated direct and inverse spaces are filled with red and blue fluids, respectively, one gets a structure schematically shown in Figure 7. Imagine now that a plumber is called in to find a leak of fluid in one of these systems; it is easy to imagine that in the absence of knowledge of how the structure is organized, the plumber might become confused. This is why such structures go under the name of ‘plumber’s nightmare’.

Figure 7 Schematic representation of the plumber’s nightmare
**Living cells and organisms**
The most potentially exciting but also challenging area where the geometry of packing and frustration come into play is in the study of living matter. Cells, for example, have carefully controlled membranes both in their periphery and surrounding the nucleus residing inside the cell. Tissues are made up of packed cells whose geometry is related to biological function and organs are built up by packing tissues in a variety of ways. Finally, tissues and organs are configured into an infinite array of different organisms in which geometry is once again critical. It is interesting to think about the familiar circulatory, respiratory, nervous and muscular circuits as *plumber's nightmares* of interpenetrating networks that are contiguous but disconnected.

The phenomena we have described above illustrate how geometry can be carefully controlled on meso and macroscales to reveal hidden principles and surprising physical and mathematical truths. By following his own artistic inspiration John Duff has similarly revealed original facets of the structure and properties of space, brought forth in pleasing and intriguing works of art.

The author is indebted to his colleague Paul Chaikin for his patient explanations of some of the subtleties of solid geometry.